Densest Subgraph in Dynamic Graph Streams

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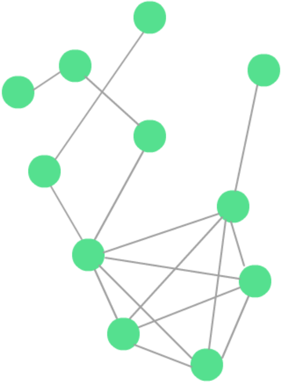
MFCS 2015

The Densest Subgraph Problem



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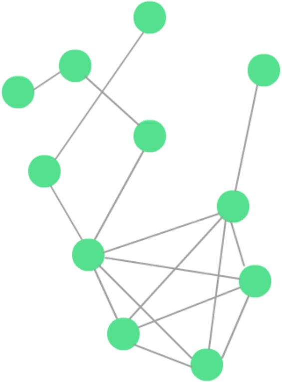

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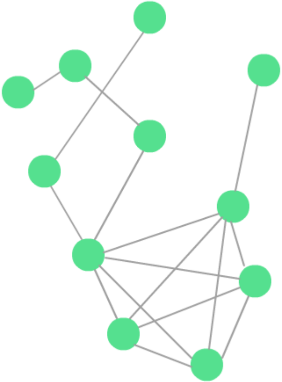

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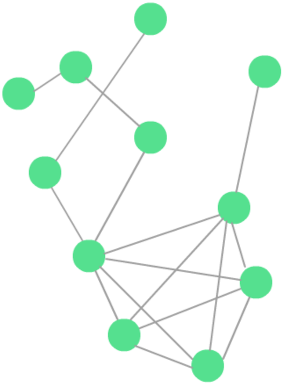

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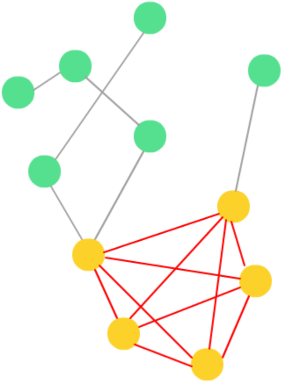

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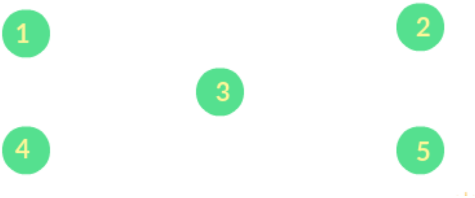

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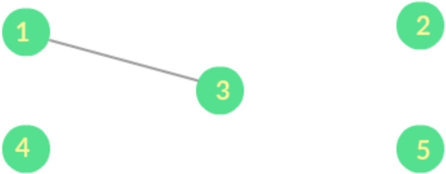

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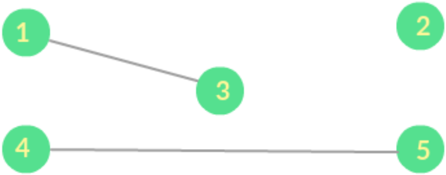

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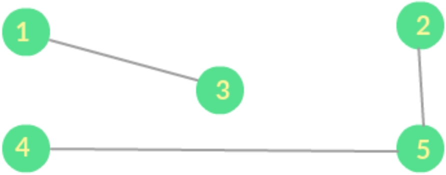

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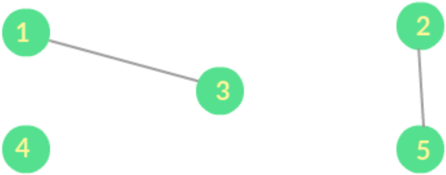

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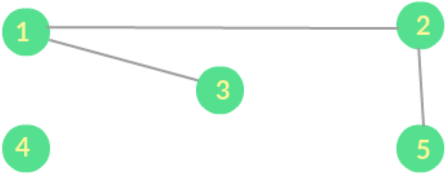

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- Prior work:
  - Bahmani et al. (PVLDB 2012) have a $(2+\epsilon)$-approximation that requires log(n) passes over the stream
  - Bhattycharya et al. (STOC 2015) only require 1 pass

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- $(1 + \epsilon)$-approximation to the densest subgraph problem
- n polylog(n) space
- polylog(n) time per update and poly(n) post-processing time
- Requires a single pass over the input stream

Densest Subgraph in Streaming

Our approach:

Edge sampling technique

Approximately preserves max density

Remaining graph has n polylog(n) edges

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ℓ_0-sampling allows emulation of edge sampling

Naive implementation is slow, but improvable

Edge Sampling Preserves Max Density

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Sample each edge in G = (V,E) with probability

$$p \approx \epsilon^{-2} \log(n) \frac{n}{m}$$

where $n = |V|$ and $m = |E|$. Call the resulting graph G'.

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Sampling Theorem

G' can be used to approximate d^* (max density of G) up to a factor $(1 + \epsilon)$.

Proof of Sampling Theorem: Preliminaries

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For any $U \subseteq V$:

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We want to show that

$$(1-\epsilon)d^* \leq \max_U \tilde{d}_U \leq (1+\epsilon)d^*$$

Proof of Sampling Theorem: Preliminaries

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Pick some $U \subseteq V$ of size k. By Chernoff, with probability $1 - n^{-9k}$,

<u>Low Density Case</u>

$$\text{if } d_U \leq \frac{d^*}{60} \text{ then } \tilde{d}_U \leq \frac{d^*}{10}$$

<u>High Density Case</u>

$$\text{if } d_U > \frac{d^*}{60} \text{ then } \tilde{d}_U \approx (1 \pm \epsilon) d_U$$

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By union bound over all U of size k, and then by all k, the above holds for all $U \subseteq V$ whp.

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Thus

$$\max_U \tilde{d_U} \geq d_{U^*} \geq (1 - \epsilon)d^*.$$

Proof of Sampling Theorem: Upper Bound

$$\max_{U} \tilde{d}_U$$

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$$\leq max\{\frac{d^*}{10}, (1+\epsilon)d_U\}$$

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Thus, putting together the upper and lower bounds,

$$(1-\epsilon)d^* \leq \max_U \tilde{d}_U \leq (1+\epsilon)d^*.$$

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We sample $r \gg mp$ edges, and when the stream is over randomly choose X $\sim$ Bin(m,p) those edges without replacement.

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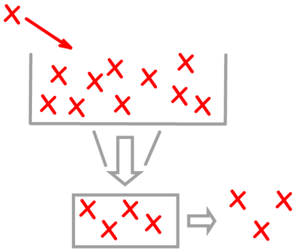

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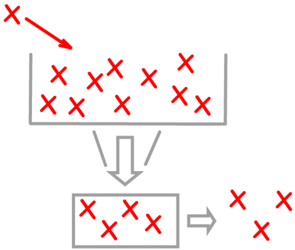

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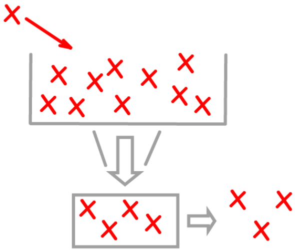

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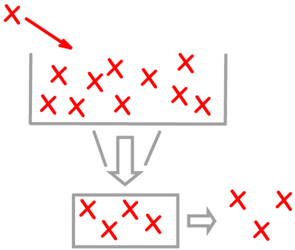

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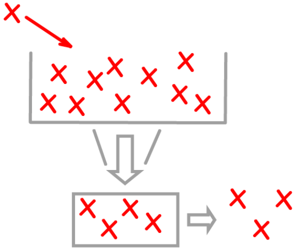

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Unfortunately, this takes $\Omega(n)$ time per update!

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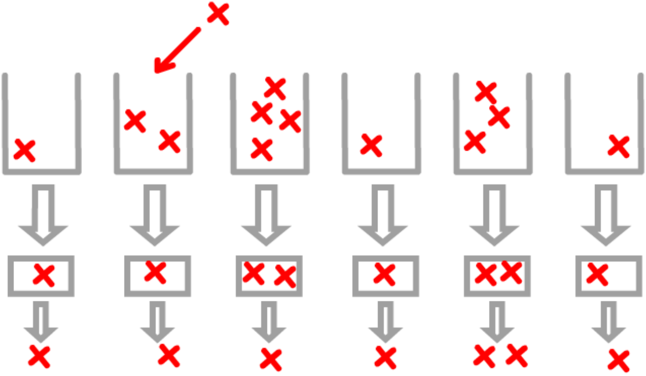
- Each time a new edge arrives, we must update every ℓ_0-sampler
- We have more than n ℓ_0-samplers, and updating each takes $O(\mathrm{polylog}\, n)$ time

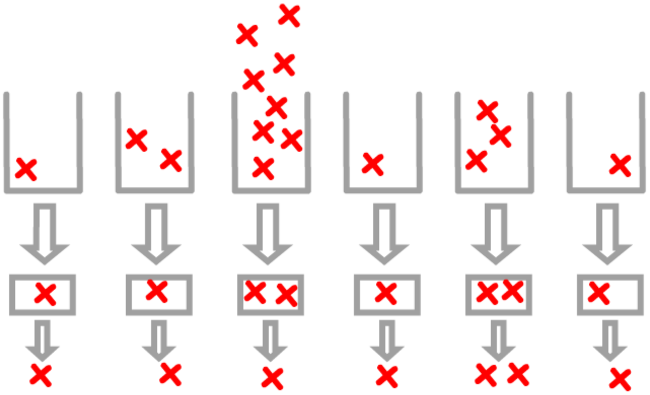
The Solution

Using a hash function, randomly partition the edge set into $\Theta(n)$ buckets. Maintain only log(n) ℓ_0-samplers for the edges in each group. When a new edge arrives in the stream, you only need to update the ℓ_0-samplers for its group!



Overflowing Buckets

Problem: Some buckets might get too full. If that happens, we can't sample those edges properly.



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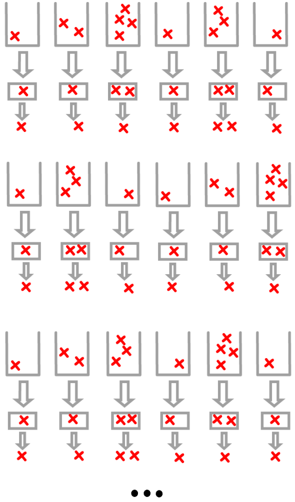

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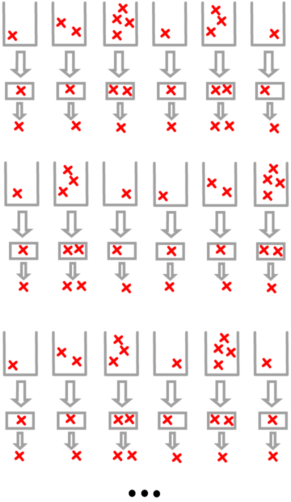

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For each edge update: $\log(n)$ partitions each with $\log(n)$ ℓ_0-samplers each with polylog(n) update time yields polylog(n) update time.

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